

A shortcut for evaluating some definite integrals from products and limits

F. M. S. Lima

Institute of Physics, University of Brasilia, P.O. Box 04455, 70919-970, Brasilia-DF, Brazil

Present address: Departamento de Fisica, Universidade Federal de Pernambuco, 50670-901, Recife-PE, Brazil

Abstract

In this short paper, I introduce an elementary method for exactly evaluating the definite integrals $\int_0^\pi \ln(\sin \theta) d\theta$, $\int_0^{\pi/2} \ln(\sin \theta) d\theta$, $\int_0^{\pi/2} \ln(\cos \theta) d\theta$, and $\int_0^{\pi/2} \ln(\tan \theta) d\theta$ in finite terms. The method consists in to manipulate the sums obtained from the logarithm of certain products of trigonometric functions at rational multiples of π , putting them in the form of Riemann sums. As this method does not involve any search for primitives, it clearly represents a good alternative to more involved integration techniques. As a bonus, I show how to apply the method for easily evaluating $\int_0^1 \ln \Gamma(x) dx$.

Keywords: Products of sines, Rational multiples of π , Definite integrals

2000 MSC: 26A06, 26A42, 08B25

Email address: fabio@fis.unb.br (F. M. S. Lima)

1. Introduction

The simplicity of the functions $\ln(\sin \theta)$, $\ln(\cos \theta)$, and $\ln(\tan \theta)$, as well as the fact they are continuous (even differentiable) except at some isolated points, suggests that the evaluation of the definite integrals $\int_0^\pi \ln(\sin \theta) d\theta$, $\int_0^{\pi/2} \ln(\sin \theta) d\theta$, $\int_0^{\pi/2} \ln(\cos \theta) d\theta$, and $\int_0^{\pi/2} \ln(\tan \theta) d\theta$ via the Fundamental Theorem of Calculus should be a straightforward task,¹ which is not true.² The usual methods for the analytic evaluation of such integrals involve advanced integration techniques, such as to expand the integrand in a power series and to integrate it term-by-term (see Sec. 11.9 of Ref. [1]), or the evaluation of a suitable contour integral on the complex plane in view to apply the Cauchy's residue theorem (see Secs. 4.1 and 4.2 of Ref. [2]). Unfortunately, these methods have some disadvantages when applied to definite integrals of “log-trig” functions. The expansion of the integrand in a Maclaurin series, e.g., is not possible for $\ln(\sin \theta)$ and $\ln(\tan \theta)$. Though this series expansion is possible for $\ln(\cos \theta)$, it is difficult to determine a closed-form expression for the general term and then to recognize the number it represents. The evaluation of a contour integral on the complex plane has the inconveniences of requiring the choice of a *suitable* integration path, usually a difficult “cosine” task, and yielding logarithms of complex (non-real) numbers and/or non-elementary transcendental functions (e.g., dilogarithm, elliptic, and hypergeometric functions), which often makes it obscure the final result as these

¹ These integrals are, in fact, *improper* because each integrand has at least one infinite discontinuity in the integration interval.

² Note that it is not possible to express the corresponding indefinite integrals in a finite, closed-form expression involving only *elementary functions*.

functions are either unknown or badly-known for most students. These inconveniences are just what one finds by appealing to mathematical softwares. For instance, Maple (release 13) and Mathematica (release 7) both return the following “stodgy” result for $\int \ln(\sin x) dx$:

$$\frac{i}{2} \text{Li}_2(e^{2ix}) + i \frac{x^2}{2} + x \ln(\sin x) - x \ln(1 - e^{2ix}),$$

where $\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is the dilogarithm function.

In this work, the integrals $\int_0^\pi \ln(\sin \theta) d\theta$, $\int_0^{\pi/2} \ln(\sin \theta) d\theta$, $\int_0^{\pi/2} \ln(\cos \theta) d\theta$, and $\int_0^{\pi/2} \ln(\tan \theta) d\theta$ are easily evaluated by taking the logarithm of certain products of trigonometric functions at rational multiples of π , which yields sums that can be written in the form of Riemann sums. The closed-form expressions emerge when we take the limit as the number of terms tends to infinity, without any search for primitives. I also show how the method, together with the reflection property of the gamma function, can be used for easily evaluating $\int_0^1 \ln \Gamma(x) dx$.

2. Some products of trigonometric functions

Let me present the products of trigonometric functions at rational multiples of π that will serve as the basis for my method.

In the Appendix A.3 of Ref. [3], in presenting an elementary proof for $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, the authors prove an identity involving $\cot^2 \theta$, θ being a rational multiple of π . From the usual rule for the product of all roots of the polynomial equation $F(x) = 0$, when applied to

$$F(x) = \sum_{j=0}^N (-1)^j \binom{2N+1}{2j+1} x^{N-j} = (2N+1) \prod_{n=1}^N (x - \cot^2 \theta_n),$$

where $\theta_n = n\pi/(2N+1)$, $n = 1, \dots, N$, they show that

$$\prod_{n=1}^N \cot^2 \theta_n = (-1)^N \frac{(-1)^N}{\binom{2N+1}{1}} = \frac{1}{2N+1}.$$

By extracting the square-root of the inverse of each side, one finds

$$\prod_{n=1}^N \tan \left(\frac{n\pi}{2N+1} \right) = \sqrt{2N+1}. \quad (1)$$

In that appendix, we also find a proof for the following identity:

$$\prod_{n=1}^{N-1} \sin \left(\frac{\pi n}{N} \right) = \frac{N}{2^{N-1}}. \quad (2)$$

From the symmetry relation $\sin(\pi/2 + \alpha) = \sin(\pi/2 - \alpha)$, it is easy to deduce that

$$\prod_{n=1}^{N-1} \sin \left(\frac{\pi n}{N} \right) = \prod_{n=1}^{\lfloor \frac{N}{2} \rfloor} \sin \left(\frac{\pi n}{N} \right) \sin \left(\frac{\pi(N-n)}{N} \right) = \prod_{n=1}^{\lfloor \frac{N}{2} \rfloor} \sin \left(\frac{\pi n}{N} \right) \sin \left(\pi - \frac{\pi n}{N} \right),$$

valid for all $N > 1$, which implies that

$$\prod_{n=1}^{\lfloor \frac{N}{2} \rfloor} \sin^2 \left(\frac{\pi n}{N} \right) = \frac{N}{2^{N-1}}. \quad (3)$$

For a product of cosines, note that, $\forall \alpha \in [0, \pi/2]$, $\sin \alpha = \cos(\pi/2 - \alpha)$. By taking $\alpha = \pi n/N$ and applying this rule on Eq. (3), one finds that

$$\prod_{n=1}^{\lfloor \frac{N}{2} \rfloor} \cos^2 \left(\frac{\pi}{2} - \frac{\pi n}{N} \right) = \frac{N}{2^{N-1}}. \quad (4)$$

These are the trigonometric products needed for evaluating the definite integrals we are interested in here.

3. Evaluation of definite integrals from products

The general idea underlying my method is to take the logarithm of a product of positive terms, convert it into a sum of logarithms and then to put this sum in the form of a Riemann sum with equally-spaced subintervals, whose limit as the number of terms tends to infinity is a definite integral. Let us apply this procedure to the products of trigonometric functions in Eqs. (1)–(4).

For instance, by taking the logarithm of each side of Eq. (2), one has

$$\sum_{n=1}^{N-1} \ln \left[\sin \left(\pi \frac{n}{N} \right) \right] = \ln N - (N-1) \ln 2. \quad (5)$$

Dividing both sides by $N-1$ gets

$$\sum_{n=1}^{N-1} \frac{\ln \left[\sin \left(\pi \frac{n}{N} \right) \right]}{N-1} = \frac{\ln N}{N-1} - \ln 2. \quad (6)$$

Now, let us define $x_n = n/N$ and $\Delta x = 1/(N-1)$. Equation (6) then reads

$$\sum_{n=1}^{N-1} \ln [\sin (\pi x_n)] \Delta x = \frac{\ln N}{N-1} - \ln 2. \quad (7)$$

Clearly, the sum at the left-hand side has the form of a Riemann sum in which the grid points x_n are equally spaced by Δx . By taking the limit as $N \rightarrow \infty$ on both sides of this equation and noting that $\lim_{N \rightarrow \infty} \frac{\ln N}{N-1} = 0$, which follows from L'Hopital rule, one has

$$\lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} \ln [\sin (\pi x_n)] \Delta x = -\ln 2, \quad (8)$$

which means that

$$\int_{0^+}^{1^-} \ln \sin (\pi x) dx = -\ln 2. \quad (9)$$

The change of variable $\theta = \pi x$ promptly yields

$$\int_0^\pi \ln \sin \theta \, d\theta = -\pi \ln 2. \quad (10)$$

When the above procedure is applied to Eq. (3) one finds that

$$2 \sum_{n=1}^{\lfloor N/2 \rfloor} \ln [\sin (\pi x_n)] \Delta x = \frac{\ln N}{N-1} - \ln 2.$$

The limit as $N \rightarrow \infty$ yields

$$2 \int_0^{\frac{1}{2}} \ln \sin (\pi x) \, dx = -\ln 2.$$

The change of variable $\theta = \pi x$ yields

$$\int_0^{\pi/2} \ln \sin \theta \, d\theta = -\frac{\pi}{2} \ln 2. \quad (11)$$

Now, from the product of cosines in Eq. (4) one has

$$2 \sum_{n=1}^{\lfloor N/2 \rfloor} \ln \left[\cos \left(\frac{\pi}{2} - \pi x_n \right) \right] \Delta x = \frac{\ln N}{N-1} - \ln 2.$$

The limit as $N \rightarrow \infty$ yields

$$2 \int_0^{\frac{1}{2}} \ln \cos \left(\frac{\pi}{2} - \pi x \right) \, dx = -\ln 2.$$

The change of variable $\theta = \frac{\pi}{2} - \pi x$ yields

$$\int_0^{\pi/2} \ln \cos \theta \, d\theta = -\frac{\pi}{2} \ln 2. \quad (12)$$

From the product of tangents in Eq. (1), one has

$$\sum_{n=1}^N \ln \left[\tan \left(\pi \frac{n}{2N+1} \right) \right] = \frac{1}{2} \ln (2N+1).$$

By substituting $2N + 1 = M$ (hence M is an odd positive integer) and then dividing both sides by M , one has

$$\sum_{n=1}^{\frac{M-1}{2}} \ln [\tan (\pi x_n)] \Delta x = \frac{\ln M}{2M},$$

where $x_n = n/M$ and $\Delta x = 1/M$. The limit as $M \rightarrow \infty$ yields

$$\lim_{M \rightarrow \infty} \sum_{n=1}^{\frac{M-1}{2}} \ln [\tan (\pi x_n)] \Delta x = \lim_{M \rightarrow \infty} \frac{\ln M}{2M} = \frac{1}{2} \lim_{M \rightarrow \infty} \frac{1}{M} = 0,$$

which means that

$$\int_0^{\frac{1}{2}} \ln \tan (\pi x) dx = 0.$$

The change of variable $\theta = \pi x$ yields

$$\int_0^{\pi/2} \ln \tan \theta d\theta = 0. \quad (13)$$

To my surprise, this method also works in evaluating the “impossible” integral $\int_0^1 \ln \Gamma(x) dx$, where $\Gamma(x)$ is the Euler gamma function.³ For this, let us add $(N - 1) \ln \pi$ on both sides of Eq. (5), which yields

$$(N - 1) \ln \pi - \sum_{n=1}^{N-1} \ln \left[\sin \left(\pi \frac{n}{N} \right) \right] = (N - 1) \ln \pi - \ln N + (N - 1) \ln 2.$$

This promptly simplifies to

$$\sum_{n=1}^{N-1} \ln \left[\frac{\pi}{\sin (\pi x_n)} \right] = (N - 1) \ln (2\pi) - \ln N. \quad (14)$$

³The adjective *impossible* certainly reflects the opinion of most undergraduate students.

Now, let us make use of the reflection property $\Gamma(x) \cdot \Gamma(1 - x) = \pi / \sin(\pi x)$, valid for all $x \notin \mathbb{Z}$. By substituting this in Eq. (14) and then dividing both sides by $N - 1$, one finds that

$$\sum_{n=1}^{N-1} \frac{\ln \Gamma(x_n) + \ln \Gamma(1 - x_n)}{N - 1} = \ln(2\pi) - \frac{\ln N}{N - 1}.$$

By taking the limit as $N \rightarrow \infty$, one finds

$$\int_0^1 [\ln \Gamma(x) + \ln \Gamma(1 - x)] dx = \ln(2\pi). \quad (15)$$

Now, note that $\int_0^1 [\ln \Gamma(x) + \ln \Gamma(1 - x)] dx = \int_0^1 \ln \Gamma(x) dx + \int_0^1 \ln \Gamma(1 - x) dx$. By substituting $y = 1 - x$ in the latter integral, one finds that the terms are identical, so

$$2 \int_0^1 \ln \Gamma(x) dx = \ln(2\pi),$$

which means that

$$\int_0^1 \ln \Gamma(x) dx = \ln \sqrt{2\pi}. \quad (16)$$

This impressive result appears on the cover of the gripping book *Irresistible integrals* [4].

I left for the reader the less obvious task of using the identity

$$\prod_{n=0}^{N-1} \sin\left(\pi \frac{n}{N} + \theta\right) = \frac{\sin(N\theta)}{2^{N-1}}, \quad (17)$$

valid for all positive integer N and all $\theta \in \mathbb{R}$, for evaluating the definite integral

$$\int_0^1 \ln |\sin(\pi x + \theta)| dx. \quad (18)$$

It may be useful to restrict yourself first to the case $\theta \neq r\pi$, with $r \in \mathbb{Q}$. The proof of the trigonometric identity in Eq. (17) is proposed as an exercise in Ref. [3] (see Ex. 6 in its Appendix A.3).

- [1] J. Stewart, *Calculus - Early Transcendentals* (6th ed.), Thomson, Belmont (USA), 2008.
- [2] M. J. Ablowitz and A. S. Fokas, *Complex Variables* (2nd ed.), Cambridge Univ. Press, New York, 2003.
- [3] I. Niven, H. S. Zuckerman, and H. L. Montgomery. *An Introduction to the Theory of Numbers* (5th ed.), Wiley, New York, 1991.
- [4] G. Boros and V. H. Moll, *Irresistible Integrals*, Cambridge Univ. Press, New York, 2004.